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Wheeler Propagator of the Lorentz Invariant Thermal Energy Propagation

Ferenc Márkus · Katalin Gambár

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Abstract Two decades ago it was made the first steps to establish a really Hamiltonian description of field theory of dissipative transport processes in order to apply all of the tools of modern field theories for these kind of processes. The great breakthrough was to introduce such scalar potential fields for the measurable quantities—temperature, pressure, etc.—, by which the complete Lagrangian-Hamiltonian formalism could be developed. Later it has been managed to create the differential equation for the relativistic invariant heat propagation by the help of the scalar field, which field can generate a dynamical temperature immediately. In the present paper it is focused on a farther step in the Lorentz invariant thermal energy propagation. Now, the temperature field satisfies a covariant field equation of a (wave-like) thermal energy propagation with finite speed—less than the speed of light. In our previous works the connection has been clarified between this scalar field and the usual local equilibrium temperature including the classical Fourier's heat conduction. The existence of a dynamical phase transition between the two kinds of propagation, between a wave and a non-wave, i.e., a dynamical phase transition between a non-dissipative and a dissipative thermal process is also found. Presently, it is pointed out that the so-called Wheeler propagator can be obtained without any difficulty for this process inspite of the existence of the dynamical phase transition and it can be seen that the causality condition is completed at the same time.

F. Márkus (✉)
Department of Physics, Budapest University of Technology and Economics, Budafoki út 8,
1521 Budapest, Hungary
e-mail: markus@phy.bme.hu

F. Márkus
e-mail: markusferi@tvn.hu

K. Gambár
Institute of Basic and Technological Sciences, Dennis Gabor College, Mérnök u. 39, 1119 Budapest,
Hungary
e-mail: gambar@gdf.hu

K. Gambár
e-mail: gakati@gmail.com

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1 Introduction

There is an old toughish problem in the theory of transport phenomena: how to resolve the question of infinite speed propagation of action in the nonequilibrium thermodynamics for those processes such as heat conduction, diffusion, etc.? This is the reason why the examination of the transition between the purely diffusive heat transfer and the wave-like ballistic heat transfer has been in the center of interest for a while. During the decades a great number of works have been published based on different examination strategy—both with the aim of Lorentz invariant and nonrelativistic approximations—to understand and solve this physical and mathematical problem. Since it is not an aim to collect and treat the historical past of this theme, thus we would like to mention some references without the completeness in time order [1–32].

The applied mathematical model is based on the Hamiltonian formulation that can include the description of both the classical heat conduction (Fourier's heat conduction) [33] and the Lorentz invariant heat propagation [34, 35] insuring a finite speed propagation of thermal signal (i.e., less than the speed of light). In the second treatment the main point is that it involves the classical heat conduction as a limit, thus we need to deal with this description in general. The Lorentz invariant description requires a Klein-Gordon type equation with a repulsive potential [33]. This repulsive interaction leads to a tachyon solution involving the so-called spinodal instability dividing the heat propagation into two parts: wave-like and diffusive. (The effect of spinodal instability is very important and often found in the modern field theories [36, 37].) Moreover, the study of thermal energy propagation from the slow heat conduction to the fast heat propagation clearly shows the existence of a dynamical phase transition [34, 38].

To introduce the reader into the concepts we shortly summarize the main steps towards the construction of Klein-Gordon equation for the present problem. We write the Hamiltonian descriptions parallel pointing out how the Lorentz invariant solution provides the classical solution.

We can formulate the relevant equations both for the wave solution with the Lagrangian L_w [33] and for the Fourier's heat conduction with the Lagrangian L_c [35]. Now, let the Lagrangians be

$$\begin{aligned} L_w &= \frac{1}{2}\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2 + \frac{1}{2c^4}\left(\frac{\partial^2\varphi}{\partial t^2}\right)^2 - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{1}{2}\frac{c^4 c_v^4}{16\lambda^4}\varphi^2, \\ L_c &= \frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + \frac{1}{2}\frac{\lambda^2}{c_v^2}\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2. \end{aligned} \tag{1}$$

Here, the field φ is a four times differentiable Lorentz invariant scalar field that generates the measurable thermal field. The speed of light is denoted by c, the heat conductivity is λ, and c_v is the specific heat. Applying the calculus of variations the corresponding Euler-Lagrange equations for the field φ can be obtained

$$\begin{aligned} 0 &= \frac{1}{c^4}\frac{\partial^4\varphi}{\partial t^4} + \frac{\partial^4\varphi}{\partial x^4} - \frac{2}{c^2}\frac{\partial^4\varphi}{\partial t^2\partial x^2} - \frac{c^4 c_v^4}{16\lambda^4}\varphi, \\ 0 &= -\frac{\partial^2\varphi}{\partial t^2} + \frac{\lambda^2}{c_v^2}\frac{\partial^4\varphi}{\partial x^4}. \end{aligned} \tag{2}$$

The above introduced scalar field is to define the physical quantities, in the present case, the temperature. Let temperature T be a Lorentz invariant dynamical temperature, and temperature $\mathcal{T}$ denotes the usual local equilibrium temperature

$$\begin{aligned} T &= \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + \frac{c^2 c_v^2}{4\lambda^2}\varphi, \\ \mathcal{T} &= -\frac{\partial\varphi}{\partial t} - \frac{\lambda}{c_v}\frac{\partial^2\varphi}{\partial x^2}. \end{aligned} \tag{3}$$

Substituting (3) into the relevant (2) we obtain two differential equations

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} - \frac{\partial^2 T}{\partial x^2} - \frac{c^2 c_v^2}{4\lambda^2}T = 0, \\ \frac{\partial\mathcal{T}}{\partial t} - \frac{\lambda}{c_v}\frac{\partial^2\mathcal{T}}{\partial x^2} = 0. \end{aligned} \tag{4}$$

The first equation is a Klein-Gordon type equation with a repulsive potential in the third term $-(c^2c_v^2/4\lambda^2)T$. The second equation pertains to the well-known classical Fourier's heat equation. By the examination of the dispersion relations it has been shown in details that this Klein-Gordon equation involves a spinodal instability, a dynamical phase transition between the wave-like and the purely dissipative thermal energy transition in the system [34, 36, 37], i.e., the classical Fourier heat conduction is involved as natural limit. The further calculations and results [38]—coupling this Lorentz invariant field with other physical fields—prove that the Lorentz invariant temperature in (3) can be considered as a really dynamical temperature.

2 Calculating the Wheeler propagator

As we can see the Lorentz invariant description involves different propagation modes. Since we would like to know more about the propagation, the transition amplitude, the completeness of causality thus in the development of the theory the next step is to find the Green function for the Klein-Gordon equation in (4) following the usual method. In the case of a differential equation D for the function Ψ

$$D\Psi(x) = F(x), \tag{5}$$

the so-called Green function $G(x,x')$ fulfills the equation

$$DG(x,x') = \delta^n(x-x'). \tag{6}$$

Here, $\delta^n(x-x')$ denotes the n dimensional Dirac-delta function, and we apply $x = (x_0 = t, x_1, x_2, \ldots, x_{n-1})$. Then the function Ψ can be expressed

$$\Psi(x) = \int F(x')G(x,x')d^n x'. \tag{7}$$

Thus the following equation should be valid for the above Green function

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} - \frac{c^2c_v^2}{4\lambda^2}G = -\delta^n(x-x'), \tag{8}$$

where the $\delta^n(x-x') = \delta^{n-1}(\mathbf{r}-\mathbf{r}')\delta(t-t')$. We express the delta function

$$\delta^n(x-x') = \frac{1}{(2\pi)^n}\int d^n k e^{ik(x-x')} \tag{9}$$

where $(k = \mathbf{k}, \omega_0)$, moreover, we introduce the d'Alembert operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \tag{10}$$

and the abbreviation

$$m^2 = \frac{c^2 c_v^2}{4\lambda^2}. \tag{11}$$

Then (8) has a simple form

$$(\Box - m^2)G = \delta^n(x-x'). \tag{12}$$

Since, the equality holds

$$(\Box - m^2)^{-1} e^{ik(x-x')} = -\frac{e^{ik(x-x')}}{k^2 - m^2}, \tag{13}$$

then we obtain

$$(\Box - m^2)^{-1}\delta^n(x-x') = -\frac{1}{(2\pi)^n}\int d^n k \frac{e^{ik(x-x')}}{k^2-m^2}. \tag{14}$$

Now, the Green function can be expressed

$$G(x,x') = \frac{1}{(2\pi)^n}\int d^n k \frac{e^{ik(x-x')}}{k^2-m^2}. \tag{15}$$

In order to calculate this integral we should find the zerus points of the denominator applying the above introduced notations

$$k^2 - m^2 = \mathbf{p}^2 - p_0^2 - m^2 = 0, \tag{16}$$

from which we obtain

$$p_0 = \pm\sqrt{\mathbf{p}^2 - m^2}. \tag{17}$$

For the later use it is important to define the propagators in a proper way. We can read out the propagator from the last term of (15)

$$G(p) = \frac{1}{\mathbf{p}^2 - p_0^2 - m^2}. \tag{18}$$

The retarded and advanced propagators [39, 40] can be expressed

$$G_{ret}(p) = G_-(p) = \left(\frac{1}{\mathbf{p}^2 - p_0^2 - m^2}\right)_{ret}, \tag{19}$$

$$G_{adv}(p) = G_+(p) = \left(\frac{1}{\mathbf{p}^2 - p_0^2 - m^2}\right)_{adv} \tag{20}$$

for tachyons due to the presence of the imaginary poles. Now, the Wheeler propagator can be expounded as a half sum of the above propagators

$$G(p) = \frac{1}{2} G_{adv}(p) + \frac{1}{2} G_{ret}(p). \tag{21}$$

In the following calculations of propagators we apply the Bochner's theorem [41–45]. The main steps of the procedure are given in logical order. It is well-known that if the function $f(x_1, x_2, \ldots, x_n)$ depends on the variable set $(x_1, x_2, \ldots, x_n)$ then its Fourier transformed is—without the factor $1/(2\pi)^{n/2}$—

$$g(y_1, y_2, \ldots, y_n) = \int d^n x f(x_1, x_2, \ldots, x_n) e^{i x_i y_i}, \tag{22}$$

where $g(y_1, y_2, \ldots, y_n)$ is the function of the variable set $(y_1, y_2, \ldots, y_n)$. However, we can introduce the variables x and y instead of the original sets

$$x = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}, \tag{23}$$

$$y = (y_1^2 + y_2^2 + \cdots + y_n^2)^{1/2}. \tag{24}$$

Now, we restrict our examinations to the spherically symmetric functions $f(x)$ and $g(y)$. The Bohner's theorem says: In these cases the above Fourier transform given by (22) can be calculated by applying the Hankel (Bessel) transformation by which we obtain

$$g(y, n) = \frac{(2\pi)^{n/2}}{y^{n/2-1}} \int_0^\infty f(x) x^{n/2} J_{n/2-1}(xy) dx. \tag{25}$$

Here, J_α is a first kind α order Bessel function. Later it will be very useful to calculate the function f with causal functions depending on the momentum space p thus we write

$$f(x, n) = \frac{(2\pi)^{n/2}}{x^{n/2-1}} \int_0^\infty g(p) p^{n/2} J_{n/2-1}(xp) dp. \tag{26}$$

It can be seen that the singularity at the origin depends on n analytically.

To obtain the Wheeler propagator, first we calculate the integral in (15) with the advanced propagator in (20)

$$\mathcal{F}\{(\mathbf{p}^2 - p_0^2 - m^2)^{-1}_{adv}\}(x) = \frac{1}{(2\pi)^n} \int d^{n-1} p e^{i\mathbf{pr}} \int_{adv} dp_0 \frac{e^{-ip_0 x_0}}{\mathbf{p}^2 - p_0^2 - m^2}. \tag{27}$$

Here, the path of integration runs parallel to the real axis and below both the poles for the advanced propagator. (In the case of the retarded propagator the path runs above the poles.) Thus, considering the propagator $G_{adv}(p)$ for $x_0 > 0$ the path is closed on the lower half plane giving null result. In the opposite case, when $x_0 < 0$, there is a non-zero finite contribution of the residues at the poles

$$p_0 = \pm\omega = \sqrt{\mathbf{p}^2 - m^2} \quad \text{if } \mathbf{p}^2 \geq m^2 \tag{28}$$

and

$$p_0 = \pm i\omega' = \sqrt{\mathbf{p}^2 - m^2} \quad \text{if } \mathbf{p}^2 \leq m^2. \tag{29}$$

After applying the Cauchy's residue theorem for the integration with respect to p_0 we obtain an $n-1$ order integral

$$\mathcal{F}\{G_{adv}\}(x) = -\frac{H(-x_0)}{(2\pi)^{n-1}} \int d^{n-1}p e^{i\mathbf{pr}} \frac{\sin[(\mathbf{p}^2 - m^2 + i0)^{\frac{1}{2}} x_0]}{(\mathbf{p}^2 - m^2 + i0)^{\frac{1}{2}}}, \tag{30}$$

where H is the Heaviside's function. Similarly, the retarded propagator can be calculated

$$\mathcal{F}\{G_{ret}\}(x) = \frac{H(x_0)}{(2\pi)^{n-1}} \int d^{n-1}p e^{i\mathbf{pr}} \frac{\sin[(\mathbf{p}^2 - m^2 + i0)^{\frac{1}{2}} x_0]}{(\mathbf{p}^2 - m^2 + i0)^{\frac{1}{2}}}. \tag{31}$$

Considering the form of the propagator in (21) and taking the propagators in (30) and (31) we obtain the Wheeler-propagator

$$\mathcal{F}\{G\}(x) = \frac{\mathrm{Sgn}(x_0)}{2(2\pi)^{n-1}} \int d^{n-1}p e^{i\mathbf{pr}} \frac{\sin[(\mathbf{p}^2 - m^2 + i0)^{\frac{1}{2}} x_0]}{(\mathbf{p}^2 - m^2 + i0)^{\frac{1}{2}}}. \tag{32}$$

To evaluate the above propagators the integrals can be rewritten by the Hankel transformation based on Bochner's theorem (25)

$$\begin{aligned} &\frac{1}{(2\pi)^{n-1}} \int d^{n-1}p e^{i\mathbf{pr}} \frac{\sin[(\mathbf{p}^2 - m^2 + i0)^{\frac{1}{2}} x_0]}{(\mathbf{p}^2 - m^2 + i0)^{\frac{1}{2}}} \\ &\quad = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \frac{1}{x^{\frac{n-1}{2}-1}} \int_0^\infty p^{\frac{n-1}{2}} \frac{\sin(\mathbf{p}^2 - m^2)^{\frac{1}{2}} x_0}{(\mathbf{p}^2 - m^2)^{\frac{1}{2}}} J_{\frac{n-1}{2}-1}(xp)\, dp, \end{aligned} \tag{33}$$

where $p = \sqrt{p_1^2 + p_2^2 + \cdots + p_{n-1}^2}$ and $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}$. The following integrals [46] are applied for the above calculations such as

$$\int_0^\infty dy\, y^{\gamma+1} \frac{\sin(a\sqrt{b^2 + y^2})}{\sqrt{b^2 + y^2}} J_\gamma(cy) = \sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\gamma} c^\gamma (a^2 - c^2)^{-\frac{1}{4}-\frac{1}{2}\gamma} J_{-\gamma-\frac{1}{2}}(b\sqrt{a^2 - c^2}), \tag{34}$$

if

$$0 < c < a, \qquad \mathrm{Re}\, b > 0, \qquad -1 < \mathrm{Re}\, \gamma < \frac{1}{2},$$

and

$$\int_0^\infty dy\, y^{\gamma+1} \frac{\sin(a\sqrt{b^2 + y^2})}{\sqrt{b^2 + y^2}} J_\gamma(cy) = 0, \tag{35}$$

if

$$0 < a < c, \qquad \mathrm{Re}\, b > 0, \qquad -1 < \mathrm{Re}\, \gamma < \frac{1}{2}.$$

In our case the parameters are

$$a = x_0, \qquad b = im = i\frac{cc_v}{2\lambda}, \qquad c = r, \qquad \gamma = \frac{n}{2} - \frac{3}{2}, \tag{36}$$

and we consider the relation between the Bessel functions

$$J_\alpha(ix) = i^\alpha I_\alpha(x), \tag{37}$$

where $I_\alpha(x)$ is the modified Bessel function. Now, we can express the advanced Wheeler function (Wheeler propagator) (30) of the tachyonic thermal energy propagation

$$W_{adv}(x) = H(-x_0)\frac{\pi}{(2\pi)^{n/2}}\left(\frac{cc_v}{2\lambda}\right)^{\frac{n}{2}-1}(x_0^2 - r^2)_+^{\frac{1}{2}(1-\frac{n}{2})} I_{1-\frac{n}{2}}\left(\frac{cc_v}{2\lambda}(x_0^2 - r^2)_+^{\frac{1}{2}}\right). \tag{38}$$

The calculation for the retarded propagator in (19) can be similarly elaborated by (31) and (33)

$$W_{ret}(x) = H(x_0)\frac{\pi}{(2\pi)^{n/2}}\left(\frac{cc_v}{2\lambda}\right)^{\frac{n}{2}-1}(x_0^2 - r^2)_+^{\frac{1}{2}(1-\frac{n}{2})} I_{1-\frac{n}{2}}\left(\frac{cc_v}{2\lambda}(x_0^2 - r^2)_+^{\frac{1}{2}}\right). \tag{39}$$

Comparing the results of (38) and (39) it can be seen that we can write one common formula easily to express the propagator. Thus the Wheeler-propagator (Green or Wheeler function) in the n dimensional space-time—remembering the construction in (21)—is

$$W(x) = \frac{\pi}{2(2\pi)^{n/2}}\left(\frac{cc_v}{2\lambda}\right)^{\frac{n}{2}-1}(x_0^2 - r^2)_+^{\frac{1}{2}(1-\frac{n}{2})} I_{1-\frac{n}{2}}\left(\frac{cc_v}{2\lambda}(x_0^2 - r^2)_+^{\frac{1}{2}}\right). \tag{40}$$

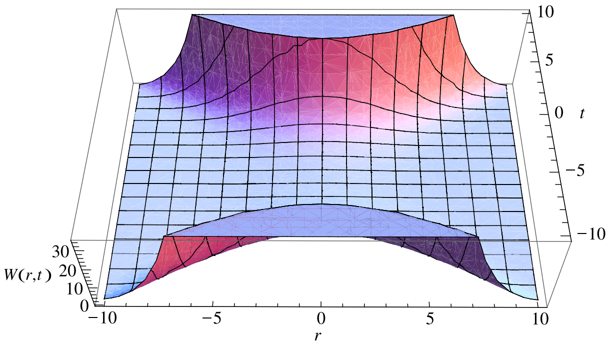
Furthermore, Bollini's and Rocca's [47–49] detailed studies show that the participating particles of the energy propagation cannot be observable directly. The tachyons do not move as free particles, thus they can be considered as the mediators of the dynamical phase transition [34].

We obtain the space 3D Wheeler propagator $[W^{(4)}]$ for the Lorentz invariant thermal energy transition if we take $n = 4$

$$W^{(4)}(r, x_0) = \frac{1}{8\pi}\left(\frac{cc_v}{2\lambda}\right)(x_0^2 - r^2)^{-\frac{1}{2}} I_{-1}\left(\frac{cc_v}{2\lambda}(x_0^2 - r^2)^{\frac{1}{2}}\right). \tag{41}$$

We plot this propagator—without the constant factor and taking the parameters $c = 1$, $c_v = 1$, $\lambda = 1$—in Fig. 1. It can be easily seen that the behavior of the propagator is in line with the causality conditions, since it differs to zero within the light cone.

Fig. 1 The Wheeler propagator in the space-time in arbitrary units



3 Summary

Following Wheeler's and Feynman's idea and Bollini's and Rocca's calculations we show the Wheeler propagator of the Lorentz invariant thermal energy propagation. The description ensures what the results also show that in spite of the negative "mass term" the propagator completes the requirement of the causality, however, the participating tachyons cannot be observed.

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